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Some theorems in Luikov's theory of heat and mass transfer in capillary-porous bodies

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Abstract—The linear theory of heat and mass transfer is considered. Some general theorems are formulated, i.e. a reciprocity theorem and a variational theorem (no use is made of the Laplace transform). The functional derived herein gives all the governing equations, including the boundary and initial conditions, as Euler equations.

INTRODUCTION

THE INTERRELATION between heat and mass transfer in porous bodies was first established by Luikov [1, 2] who proposed a two-term relationship for non-isothermal mass diffusion and also determined experimentally the coefficients of diffusion and thermo-diffusion for a number of moist materials. Later [3, 4] via the use of thermodynamics of irreversible processes, he defined a coupled system of partial differential equations for heat and mass transfer potential distributions in porous bodies. Applications in this and other fields such as drying theory, building thermo-physics and heat and moisture migration in soils can be found in ref. [5]. Independently, Krischer [6] and De Vries [7] also proposed systems of differential equations of the Luikov type for temperature and moisture content distributions in porous bodies.

The analytical solution of these types of equations presents great mathematical difficulties, and consequently solutions are given for only the simplest of geometrical configurations and boundary conditions [4]. In any realistic problem resort must be made to numerical techniques. These have usually been based on some general theorems, i.e. a reciprocity theorem and a variational theorem.

The variational formulation based on local potential to simplified non-linear heat and mass transfer equations was proposed by Kumar [8]. Glazunov [9] proved that for the non-linear transport problem variational classic type theorems do not exist. Some non-classic type principles for solution of the non-linear interrelated heat and mass transfer problems are available in refs. [10–12].

In this paper Luikov's linear theory of heat and mass transfer is considered and the reciprocity theorem and the variational theorem are established. The procedure shown herein does not require any transformation of the field equations and includes the boundary and initial conditions.

TRANSFER EQUATIONS

The governing equations in the linear theory of heat and mass transfer (Luikov's equations) are [3–5]

$$\frac{\partial T}{\partial t} = \left(a + \frac{\varepsilon r a_m \delta}{c}\right)\Delta T + \frac{\varepsilon r a_m}{c}\Delta W$$

$$\frac{\partial W}{\partial t} = a_m(\Delta W + \delta \Delta T), \quad x \in B, \quad t > 0 \tag{1}$$

where T denotes the temperature, W the mass content, a the thermal diffusivity, a_m the mass diffusivity, ε the phase-change criterion (i.e. $\varepsilon = 1$ all vapour, $\varepsilon = 0$ all liquid), r the latent heat of evaporation, c the specific heat, δ the thermogradient coefficient, x the spatial position, t the time, and Δ the Laplace operator.

To the above field equations one adjoining the boundary conditions

$$q \equiv -\lambda\frac{\partial T}{\partial n} - \varepsilon r a_m \rho\left(\frac{\partial W}{\partial n} + \delta\frac{\partial T}{\partial n}\right) = \hat{q}(x,t), \quad x \in \partial B_q,\ t > 0 \tag{2}$$

$$j \equiv -a_m\rho\left(\frac{\partial W}{\partial n} + \delta\frac{\partial T}{\partial n}\right) = \hat{j}(x,t), \quad x \in \partial B_j,\ t > 0 \tag{3}$$

$$T = \hat{T}(x,t), \quad x \in \partial B_T,\ t > 0 \tag{4}$$

$$W = \hat{W}(x,t), \quad x \in \partial B_W,\ t > 0 \tag{5}$$

and the initial conditions

$$T = T_0(x), \quad x \in \bar{B},\ t = 0 \tag{6}$$

$$W = W_0(x), \quad x \in \bar{B},\ t = 0 \tag{7}$$

where λ is the thermal conductivity, ρ the density of a perfectly dry body, $\mathbf{n}$ the unit outward normal, $\partial/\partial n$ the normal derivative; $\hat{T}$ denotes the prescribed temperature on ∂B_T, $\hat{W}$ the moisture content on ∂B_W, $\hat{q}$

NOMENCLATURE

a	thermal diffusivity [$m^2 s^{-1}$]
a_m	mass diffusivity [$m^2 s^{-1}$]
$\mathbf{A}$	matrix operator
B	body
∂B	boundary
c	specific heat [$J kg^{-1} K^{-1}$]
E	space
f	functional
$\mathbf{f}$	vector
j	mass flux [$kg m^2 s^{-1}$]
$\mathbf{n}$	unit outward normal
q	heat flux [$W m^{-2}$]
r	heat of phase change [$J kg^{-1}$]
t	time [s]
T	temperature [K]
$\mathbf{u}$	vector
W	moisture content [$kg kg^{-1}$]
x	spatial position [m].

Greek symbols

δ	thermogradient coefficient [K^{-1}]
ε	phase-change criterion
λ	thermal conductivity [$W m^{-1} K^{-1}$]
ρ	density [$kg m^{-3}$].

Subscripts and superscripts

α, β	system
$\hat{}$	surface
$'$	dual
0	initial
E	space
T, q	heat
T	transposition
W, j	mass.

Operators

∇	gradient
Δ	Laplacian
$*$	convolution
$\cdot$	scalar product
$\langle \cdot, \cdot \rangle$	bilinear form.

the heat flux on ∂B_q, $\hat{j}$ the mass flux on ∂B_j; T_0 and W_0 are prescribed initial values; $\bar{B} = B \cup \partial B$.

The set of basic equations (1)–(7) can be put into the operator form

$$\mathbf{Au} + \mathbf{f} = \mathbf{0}, \quad \mathbf{A}: E \to E', \mathbf{u} \in E, \mathbf{f} \in E' \tag{8}$$

where

$$\mathbf{u} = [T, W; T, W; T, W, q, j]^{\mathrm{T}} \tag{9}$$

$$\mathbf{f} = [0, 0, -\delta c \rho T_0, -\varepsilon r \rho W_0; \delta \hat{q}, \varepsilon r \hat{j}, -\delta \hat{T}, -\varepsilon r \hat{W}]^{\mathrm{T}} \tag{10}$$

and $\mathbf{A}$ is a linear matrix (8×8) operator the non-zero elements of which are given by

$$\begin{aligned}
A_{11} &= \delta c \rho \left(\frac{\partial}{\partial t} - \left(a + \frac{\varepsilon r a_m \delta}{c} \right) \Delta \right) \\
A_{12} &= -\varepsilon r a_m \rho \Delta = A_{21} \\
A_{22} &= \varepsilon r \rho \left(\frac{\partial}{\partial t} - a_m \Delta \right) \\
A_{33} &= \delta c \rho \\
A_{44} &= \varepsilon r \rho \\
A_{57} &= -\delta = -A_{75} \\
A_{68} &= -\varepsilon r = -A_{86}.
\end{aligned} \tag{11}$$

Here E denotes the space of ordered arrays of the form given in equation (9). The dual space of E is denoted by E'.

RECIPROCITY THEOREM

Alternatively, one may consider a problem, equivalent to equation (8) given by [13]

$$\langle \mathbf{Au}, \mathbf{v} \rangle_E = \langle \mathbf{f}, \mathbf{v} \rangle_E, \quad \mathbf{v} \in E \tag{12}$$

where $\langle \cdot, \cdot \rangle_E$ denotes the bilinear form on $E \times E'$ represented by the integrals

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{v} \rangle_E = &\int_B (f_1 * v_1 + f_2 * v_2 + f_3(x, 0) v_3 \\
&+ f_4(x, 0) v_4)\, \mathrm{d}V + \int_{\partial B_q} (f_5 * v_5)\, \mathrm{d}S + \int_{\partial B_j} (f_6 * v_6)\, \mathrm{d}S \\
&+ \int_{\partial B_T} (f_7 * v_7)\, \mathrm{d}S + \int_{\partial B_W} (f_8 * v_8)\, \mathrm{d}S
\end{aligned} \tag{13}$$

where

$$\mathbf{f} = [f_1, \ldots, f_8]^{\mathrm{T}}, \quad \mathbf{v} = [v_1, \ldots, v_8]^{\mathrm{T}}.$$

In equation (13) $f * v$ denotes the convolution of f and v [14]

$$f * v(x, t) = \int_0^t f(x, t - \tau) v(x, \tau)\, \mathrm{d}\tau. \tag{14}$$

The capillary-porous body is considered subject to two different systems of heat and mass loadings $\mathbf{f}^\alpha$ and two corresponding configurations $\mathbf{u}^\alpha$, where $\alpha = 1, 2$.

Theorem 1. If a capillary-porous body is subjected to two different systems of heat and mass loadings $\mathbf{f}^\alpha$, then between the corresponding configurations $\mathbf{u}^\alpha$ there is the following relation:

$$\langle \mathbf{f}^1, \mathbf{u}^2 \rangle_E = \langle \mathbf{f}^2, \mathbf{u}^1 \rangle_E \tag{15}$$

where

$$\langle \mathbf{f}^\alpha, \mathbf{u}^\alpha \rangle_E = \int_B (-\delta c \rho T^\alpha T_0^\beta - \varepsilon r \rho W^\alpha W_0^\beta)\,\mathrm{d}V + \int_{\partial B_q} (\delta T^\alpha * \hat{q}^\beta)\,\mathrm{d}s + \int_{\partial B_j} (\varepsilon r W^\alpha * \hat{j}^\beta)\,\mathrm{d}S - \int_{\partial B_T} (\delta q^\alpha * \hat{T}^\beta)\,\mathrm{d}S - \int_{\partial B_W} (\varepsilon r j^\alpha * \hat{W}^\beta)\,\mathrm{d}S,$$

$$\alpha, \beta = 1, 2, \alpha \neq \beta. \tag{16}$$

Proof. On the basis of relation (12) one has

$$\langle \mathbf{A}\mathbf{u}^\alpha, \mathbf{u}^\beta \rangle_E = \langle \mathbf{f}^\alpha, \mathbf{u}^\beta \rangle_E, \quad \alpha, \beta = 1, 2, \alpha \neq \beta. \tag{17}$$

From equations (9)–(11), (13) and (17) (using Green's theorem and properties of the convolution), one obtains

$$\langle \mathbf{A}\mathbf{u}^1, \mathbf{u}^2 \rangle_E = \rho \int_B \left(\delta c \left(\frac{\partial}{\partial t}(T^1 * T^2) + \left(a + \frac{\varepsilon r a_m \delta}{c} \right) \nabla T^1 * \cdot \nabla T^2 \right) + \varepsilon r a_m \delta (\nabla T^1 * \cdot \nabla W^2 + \nabla W^1 * \cdot \nabla T^2) + \varepsilon r \left(\frac{\partial}{\partial t}(W^1 * W^2) + a_m \nabla W^1 * \cdot \nabla W^2 \right) \right) \mathrm{d}V + \int_{\partial B_T} (\delta (q^2 * T^1 + q^1 * T^2))\,\mathrm{d}S + \int_{\partial B_W} (\varepsilon r (j^2 * W^1 + j^1 * W^2))\,\mathrm{d}S \tag{18}$$

which implies

$$\langle \mathbf{A}\mathbf{u}^1, \mathbf{u}^2 \rangle_E = \langle \mathbf{A}\mathbf{u}^2, \mathbf{u}^1 \rangle_E. \tag{19}$$

From equations (17) and (19) one obtains relation (15). This completes the proof of the theorem.

VARIATIONAL THEOREM

The necessary and sufficient condition that there exists a variational functional corresponding to the operator, equation (8), is that equation (19) holds for each $\mathbf{u}^1, \mathbf{u}^2 \in E$, i.e. the bilinear form must be symmetric in $\mathbf{u}^1$ and $\mathbf{u}^2$ [15].

The corresponding variational functional is given by

$$f(\mathbf{u}) = \tfrac{1}{2}\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle_E + \langle \mathbf{f}, \mathbf{u} \rangle_E. \tag{20}$$

A demonstration of this important result can be found in ref. [16].

From equations (9)–(11), (13) and (20) (using Green's theorem and the property of the convolution) one obtains

$$f(\mathbf{u}) \equiv f(T, W) = \rho \int_B \left(\tfrac{1}{2}\delta c \left(\frac{\partial}{\partial t}(T * T) + \left(a + \frac{\varepsilon r a_m \delta}{c} \right) \nabla T * \cdot \nabla T + \varepsilon r a_m \delta \nabla T * \cdot \nabla W + \tfrac{1}{2}\varepsilon r \left(\frac{\partial}{\partial t}(W * W) + a_m \nabla W * \cdot \nabla W \right) - \delta c T_0 T - \varepsilon r W_0 W \right) \right) \mathrm{d}V + \int_{\partial B_q} (\delta \hat{q} * T)\,\mathrm{d}S + \int_{\partial B_j} (\varepsilon r \hat{j} * W)\,\mathrm{d}S + \int_{\partial B_T} (\delta (T - \hat{T}) * q)\,\mathrm{d}S + \int_{\partial B_W} (\varepsilon r (W - \hat{W}) * j)\,\mathrm{d}S. \tag{21}$$

Thus, $f(\mathbf{u})$ in equation (21) is the functional associated with $\mathbf{A}\mathbf{u} + \mathbf{f}$ in equation (8); that is, the solution of the set of equations (1)–(7) is a critical point (a point $\mathbf{u} \in E$ is called a critical point of the functional $f(\mathbf{u})$ if $\operatorname{grad} f = \mathbf{0}$) of f. This is proved in the following theorem.

Theorem 2. Let $\mathbf{u} \in E$, and let $f(\mathbf{u})$ have a linear Gâteaux differential at every $\mathbf{u}$, where f is the functional defined in equation (21). Then

$$\delta f(\mathbf{u}, \mathbf{u}') = 0 \tag{22}$$

if and only if $\mathbf{u}$ is a solution of equation (8).

Proof. Let $\mathbf{u}'$ be an arbitrary element in E. Then the Gâteaux differential of f is

$$\delta f(\mathbf{u}, \mathbf{u}') = \langle \operatorname{grad} f(\mathbf{u}), \mathbf{u}' \rangle_E = \rho \int_B \left(\delta c \left(\frac{\partial T}{\partial t} - \left(a + \frac{\varepsilon r a_m \delta}{c} \right) \Delta T - \frac{\varepsilon r a_m}{c} \Delta W \right) * T' + \varepsilon r \left(\frac{\partial W}{\partial t} - a_m (\Delta W + \delta \Delta T) \right) * W' + \delta c (T(0) - T_0) T' + \varepsilon r (W(0) - W_0) W' \right) \mathrm{d}V + \int_{\partial B_q} (\delta (\hat{q} - q) * T')\,\mathrm{d}S + \int_{\partial B_j} (\varepsilon r (\hat{j} - j) * W')\,\mathrm{d}S + \int_{\partial B_T} (\delta (T - \hat{T}) * q')\,\mathrm{d}S + \int_{\partial B_W} (\varepsilon r (W - \hat{W}) * j')\,\mathrm{d}S = \langle \mathbf{A}\mathbf{u} + \mathbf{f}, \mathbf{u}' \rangle_E. \tag{23}$$

One first proves sufficiency. Suppose that $\mathbf{u} \in E$ is a solution of equations (1)–(7). Then equation (23) becomes

$$\langle \mathbf{A}\mathbf{u} + \mathbf{f}, \mathbf{u}' \rangle_E = \langle \operatorname{grad} f(\mathbf{u}), \mathbf{u}' \rangle_E = \mathbf{0} \tag{24}$$

which implies equation (22).

To prove the necessity, assume that equation (22) holds. In view of Lemmas 1–4 [17] one can see that

$$\operatorname{grad} f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{f} = \mathbf{0}. \tag{25}$$

This completes the proof of the theorem.

CONCLUSIONS

The results obtained in this paper can make a base to construct the numerical solutions of initial-boundary value problems of linear heat and mass transfer. The variational formulation for the finite element method [18] and the reciprocity equation for the boundary element method [19]. The fundamental solutions to this equation, corresponding to a concentrated heat and mass source, can be found in ref. [20].

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QUELQUES THEOREMES DANS LA THEORIE DE LUIKOV SUR LE TRANSFERT DE CHALEUR ET DE MASSE DANS LES CORPS MICROPOREUX

Résumé—On considère la théorie linéaire du transfert de chaleur et de masse. Quelques théorèmes généraux sont formulés tels que, par exemple, un théorème de réciprocité et un théorème variationnel (il n'est pas fait usage de la transformation de Laplace). La fonctionnelle utilisée donne toutes les équations fondamentales qui incluent les conditions aux limites et initiales, comme les équations d'Euler.

EINIGE THEOREME ZUR LUIKOV'SCHEN THEORIE DES WÄRME- UND STOFFTRANSPORTS IN KAPILLAR-PORÖSEN KÖRPERN

Zusammenfassung—Die lineare Theorie des Wärme- und Stofftransports wird betrachtet. Einige allgemeingültige Theoreme werden formuliert, z. B. ein Reziprozitätstheorem und ein Variationstheorem (die Laplace-Transformation wird nicht verwendet). Das hier hergeleitete Funktional ergibt alle maßgebenden Gleichungen sowie die Rand- und Anfangsbedingungen und die Euler-Gleichungen.

НЕСКОЛЬКО ТЕОРЕМ ТЕОРИИ ТЕПЛО- И МАССОПЕРЕНОСА ЛЫКОВА В КАПИЛЛЯРНО-ПОРИСТЫХ ТЕЛАХ

Аннотация—Рассматривается линейная теория тепло- и массопереноса. Сформулировано несколько общих теорем, а именно, теорема взаимности и вариационная теорема (без использования преобразования Лапласа). Выведенный функционал дает возможность получить все основные уравнения типа уравнений Эйлера с граничными и начальными условиями.